Nonclassical Solutions of Fully Nonlinear Elliptic Equations

Nikolai Nadirashvili,[*] Serge Vlăduţ[†]

Abstract. We prove the existence of non-smooth solutions to fully nonlinear elliptic equations.

1   Introduction

In this paper we study the regularity of solutions of fully nonlinear elliptic equations of the form

$$F(D^2 u) = 0 \tag{1}$$

defined in a domain of $\mathbf{R}^n$. Here $D^2 u$ denotes the Hessian of the function u. We assume that F is uniformly elliptic, i.e. there exists a constant $\Lambda \geq 1$ (called an *ellipticity constant*) such that

$$\Lambda^{-1}|\xi|^2 \leq F_{u_{ij}}\xi_i\xi_j \leq \Lambda|\xi|^2 \,, \forall \xi \in \mathbf{R}^n \,. \tag{2}$$

Here, u_{ij} denotes the partial derivative $\partial^2 u/\partial x_i \partial x_j$. A function u is called a *classical* solution of (1) if $u \in C^2(\Omega)$ and u satisfies (1). Actually, any classical solution of (1) is a smooth ($C^{\alpha+3}$) solution, provided that F is a smooth (C^α) function of its arguments.

Consider the following Dirichlet problem

$$\begin{cases} F(D^2 u) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \,, \end{cases} \tag{3}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ and φ is a continuous function on $\partial\Omega$.

It is not difficult to prove that problem (3) has no more than one classical solution (see e.g. [GT]). The basic problem is the existence of such classical solutions. Although the first systematic study of the Dirichlet problem for fully

[*]LATP, CMI, 39, rue F. Joliot-Curie, 13453 Marseille FRANCE, nicolas@cmi.univ-mrs.fr
[†]IML, Luminy, case 907, 13288 Marseille Cedex FRANCE, vladut@iml.univ-mrs.fr

nonlinear equations was done by Bernstein at the beginning of the 20-th century (see [GT]), the first complete result didn't appear until 1953, when Nirenberg proved the existence of a classical solution to problem (3) in dimension $n = 2$ ([N]). For $n \geq 3$, the problem of the existence of classical solutions to Dirichlet problem (3) remained open.

In order to get a solution to the problem (3) one can try to extend the notion of the classical solution of the equation (1). That was done recently: Crandall-Lions and Evans developed the concept of viscosity (weak) solutions of the fully nonlinear elliptic equations. As a characteristic property for such extension can be taken the maximum principle in the following form:

Let u_1, u_2 be two solutions of the following equations, $F(D^2 u_1) = f_1$ in Ω and $F(D^2 u_2) = f_2$ in Ω. Then for any subdomain $G \subset \Omega$ the inequalities $f_1 \leq f_2$ $(f_1 \geq f_2)$ in G and $u_1 \geq u_2$ $(u_1 \leq u_2)$ on ∂G imply the inequality $u_1 \geq u_2$ $(u_1 \leq u_2)$ in G.

Such maximum principle holds for C^2 functions u_1, u_2. We call a continuous function u_1 a viscosity solution of $F(D^2 u_1) = f_1$ if the above maximum principle holds for u_1 and all C^2-functions u_2.

It is possible to prove the existence of a viscosity solution to the Dirichlet problem (3) and Jensen's theorem says that the viscosity solution of the problem (3) is unique. For more details see [CC], [CIL].

There are important classes of the fully nonlinear Dirichlet problems for which the viscosity solution is in fact a classical one, e.g., due to Krylov-Evans regularity theory, in the case when the function F is convex, (see [CC], [K]). However, for the general F the problem of the coincidence of viscosity solutions with the classical remained open.

The central result of this paper is the existence of nonclassical viscosity solution of (1) in the dimension 12. More precisely we prove

Theorem. *The function*

$$w(x) := \frac{Re(\omega_1 \omega_2 \omega_3)}{|x|},$$

where $\omega_i \in \mathbf{H}$, $i = 1, 2, 3$, are Hamiltonian quaternions, $x = (\omega_1, \omega_2, \omega_3) \in \mathbf{H}^3 = \mathbf{R}^{12}$ is a viscosity solution in $\mathbf{R}^{12}$ of a uniformly elliptic equation (1) with a smooth F.

One can find the explicit expression for w in the coordinates of $\mathbf{R}^{12}$ in Sections 3 and 4. The elliptic operator F will be defined in a constructive way in Section 2, and its ellipticity constant $\Lambda < 10^8$.

As an immediate consequence of the theorem we have

Corollary. *Let $\Omega \subset \mathbf{R}^{12}$ be the unit ball and $\varphi = w$ on $\partial \Omega$. Then there exists a smooth uniformly elliptic F such that the Dirichlet problem (2) has no classical solution.*

Homogeneous order 2 function w is smooth in $\mathbf{R}^{12}\backslash\{0\}$ and has discontinuous second derivatives at 0. It is interesting to notice that the set of homogeneous order $\alpha \in \mathbf{R}$ solutions of (1) in $\mathbf{R}^n \setminus \{0\}$ for $\alpha \neq 2$ has a simple structure: each such solution of (1) has to be also a solution of a linear elliptic equation with constant coefficients, [NY].

The question on the minimal dimension n for which there exist nontrivial homogeneous order 2 solutions of (1) remains open. We notice that from the result of Alexsandrov [A] it follows that any homogeneous order 2 solution of the equation (1) in $\mathbf{R}^3$ with a real analytic F should be a quadratic polynomial. For a smooth and less regular F similar results in the dimension 3 one can find in [HNY].

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2   The Hessian Problem

2.1

Let w be a homogeneous function of order 2, defined on $\mathbf{R}^n$ and smooth in $\mathbf{R}^n \setminus \{0\}$. Then the Hessian of w is homogeneous of order 0, and defines a map

$$H : S^{n-1} \to D^2 w \in Q \ ,$$

where Q denotes the space of quadratic forms on $\mathbf{R}^n$, which we will sometimes identify as symmetric $n \times n$ matrices, $Q \simeq S^{n \times n}$. The inner product of $a, b \in Q$ is given by $a \cdot b = trace(ab)$. We say that w *satisfies property* (H) (w is a solution of the Hessian Problem) if the following holds:

(H) 1). *The map $H : S^{n-1} \to Q$ is a smooth embedding.*

2). *There exists a constant $M \geq 1$ such that for any two points $a, b \in H(S^{n-1}), a \neq b$, if $\mu_1 \geq \cdots \geq \mu_n$ denote the eigenvalues of the quadratic form $a - b$, then*

(4) $$1/M < -\mu_1/\mu_n < M \ .$$

Main Lemma. *If function w satisfies hypotheses* (H) *then w is a viscosity solution in $\mathbf{R}^n$ of a uniformly elliptic equation* (1) .

2.2

Let us choose in the space

Q an orthonormal coordinate system $z_1, \ldots, z_k, s$, $k = \frac{n(n+1)}{2} - 1$ such that $\sqrt{n}s$ is the trace. Let $\pi : Q \to Z$ be the orthogonal projection of Q onto the z-space. For $\lambda \geq 1$, we denote by K_λ the cone

$$K_\lambda = \{a \in Q : \text{ there exists } C > 0 \text{ s. t. the eigenvalues of } a \in [C/\lambda, C\lambda] \,\}.$$

Notice, that inequalities (2) mean that the eigenvalues of ∇F are on the segment $[\Lambda^{-1}, \Lambda]$. In particular (2) implies the inclusion $\nabla F \in K_\Lambda$. Since $0 \notin K_\Lambda$ it follows in particular that $F^{(-1)}(0) \in C^\infty$.

Since on Q the maximal eigenvalue of a quadratic form is a convex function and the minimal eigenvalue is a concave function it follows that K_Λ is a convex cone.

Let K_λ^* denote the adjoint cone of K_λ, that is,

$$K_\lambda^* = \{b \in Q : b \cdot c \geq 0 \text{ for all } c \in K_\lambda\} \,.$$

As an adjoint to a convex cone the cone K_λ^* is a convex cone itself.

Set $L_\lambda = Q \setminus (K_\lambda^* \cup -K_\lambda^*)$. Notice that $a \in L_\lambda$ is equivalent to $a \cdot b = 0$ for some $b \in K_\lambda$, i.e., L_λ is a union of all hyper-planes in Q with normals in K_λ. Since the quadratic forms of K_λ are positively defined it follows that the vector $I \in K_\lambda^*$. Let $K \subset Q$ be a cone with a smooth strictly convex base such that $K_{2\lambda} \subset K \subset K_\lambda$. Let $e_1, \ldots e_k, I$ be an orthonormal basis of Q corresponding to the coordinates $z_1, \ldots, z_k, s$. Then any matrix $b \in Q$ can be written as

$$b = sI + \sum_{j=1}^{k} z_j e_j.$$

Now define

$$x(z) := \inf\{c : (a + cI) \in K^*\}$$

for $a := \sum_{j=1}^{k} z_j e_j$. The graph of the function $s = x(z)$ represents the boundary of the cone K. Clearly $x(\cdot)$ is Lipshitz, convex, homogenous, smooth outside the origin and $x(0) = 0$. By a simple computation we get that $|\nabla x| < \sqrt{n}$.

Let $G \subset Q$ be a set. We say that G *satisfies the λ-cone condition* if for any two points $a, b \in G$, the matrix $a - b \in L_\lambda$.

Lemma. *Let $\Sigma \subset Q$ be a smooth compact $(n-1)$-dimensional manifold. Assume that Σ satisfies the λ-cone condition. Then there exists a smooth function F on Q such that $F(\Sigma) = 0$, and which satisfies the inequality (2) with the ellipticity constant $\Lambda < 4\lambda^2 \sqrt{n}$.*

Proof of the lemma. Set $\sigma = \pi(\Sigma)$. We prove that Σ is a graph of a Lipschitz continuous function,

$$\Sigma = \{z \in \sigma : s = g(z)\} \,.$$

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Let $a, \hat{a} \in \Sigma$, $a = sI + \sum_{j=1}^{k} z_j e_j$, $\hat{a} = \hat{s}I + \sum_{j=1}^{k} \hat{z}_j e_j$. Since $a - \hat{a} \in L_\lambda$, we have $-x(z - \hat{z}) \leq \hat{s} - s \leq x(z - \hat{z})$. Since $x(0) = 0$, $g(z) := s$ is single-valued. Also $|g(z) - g(\hat{z})| = |s - \hat{s}| \leq |x(z - \hat{z})| \leq C|z - \hat{z}|$.

Hence, since Σ is a smooth surface g is a smooth function and σ is a smooth surface as well.

Let G_k^m be the Grassmannian manifold of m-dimensional subspaces of the k-dimensional subspace z of Q. Let $l \in G_k^{n-1}$ and $t : l \to s$ be a linear function on l, such that the graph of t satisfies the λ-cone condition. All such linear functions t defined on all $l \in G_k^{n-1}$ we denote by τ. Let $t \in \tau$ defined on $l \in G_k^{n-1}$ be such that $|\nabla t| \neq 0$. Then there exists a constant $c' > 1$ such that $c't \leq x$ on l and there is a point $z' \in l$, $|z'| = 1$ with $c't(z') = x(z')$. Since K^* is a strictly convex cone the vector z' is unique. Denote

$$\eta(t) = \{z \in l, t(z) = 0\}.$$

Then $\eta(t) \perp \nabla t$. Since $\eta(t) \subset Z$ and $\eta(t)$ is tangent to the cone K^* at $z', x(z')$ it follows that $\eta(t) \perp \nabla x(z')$.

Let θ be a smooth function defined on $[1, \infty)$ such that $0 \leq \theta \leq 1$, $\theta = 1$ on $[1, A]$, $\theta = 0$ on $[2A, \infty)$, where a sufficiently large constant A will be chosen later. Set
$$\nu(t) = \theta(c')\nabla x(z') + (1 - \theta(c'))\nabla t.$$

For $z \in \sigma$ we denote by $l(z) \in G_k^{(n-1)}$ the tangent subspace to σ at z. Let $t_z(x)$, $x \in l(z)$, be the differential of g at z.

Let $z \in \sigma$. Denote by $\Psi(z)$ $(n-1)$-dimensional subspace spanned by $\eta(t_z)$ and $\nu(t_z)$, if $\nabla t_z \neq 0$. If $\nabla t_z = 0$ we set $\Psi(z) = l$. Thus we defined a smooth map
$$\Psi : \sigma \to G_k^{n-1}.$$

There exists $\gamma \subset Z$ a closed neighborhood of σ such that γ is diffeomorphic to $\sigma \times B$, where B is the $(k-n+1)$-dimensional disk. We define a projection $\gamma \to \sigma$ such that the fiber $p^{-1}(z)$ is orthogonal to $\Psi(z)$ at $z \in \sigma$. Since $(\nabla x(z'), z') > 0$ the fiber $p^{-1}(z)$ is transversal to σ at z. We extend the function g to γ by $g(y) = g(p(y))$. Let Γ be the graph of g over γ. Let $z \in \sigma$ and $dg(z)$ be the differential of g over γ. For sufficiently large constant A the following alternative holds: either $|\nabla g(z)|$ is sufficiently small, or the graph of $c'dg(z)$ is tangent to the cone K^*. In both cases the graph of $dg(z)$ satisfies 2λ-cone condition. Since $g \in C^1(Z)$ and along the fibers the function g is a constant, we may assume the neighborhood γ to be sufficiently small so that Γ satisfies the λ-cone condition.

Since $K_{2\lambda} \subset K$ $g \in C^1(\sigma)$ the function g has an extension $\widetilde{g}$ from the set γ to $\mathbf{R}^k$ such that $\widetilde{g}$ is a Lipschitz function and the graph of $\widetilde{g}$ satisfies the 2λ-cone condition. One can define such an extension $\widetilde{g}$ simply by the formula

$$\widetilde{g}(z) := \inf_{w \in \gamma} \{g(w) + x(z - w)\}.$$

To demonstrate that this formula works let $(z, \tilde{g}(z))$, $(\hat{z}, \tilde{g}(\hat{z}))$ lie on the graph $\tilde{g}$. We must show

$$-x(z - \hat{z}) \leq \tilde{g}(z) - \tilde{g}(\hat{z}) \leq x(z - \hat{z}).$$

Now

$$\tilde{g}(\hat{z}) = g(w) + x(\hat{z} - w)$$

for some $w \in \gamma$. Thus

$$\tilde{g}(z) - \tilde{g}(\hat{z}) \leq g(w(\hat{z})) + x(z - w(\hat{z})) - (g(w(\hat{z})) + x(\hat{z} - w(\hat{z}))) \leq x(z - \hat{z}),$$

since $x(a + b) \leq x(a) + x(b)$, as $x(\cdot)$ is convex and homogenous. Similarly

$$\tilde{g}(z) - \tilde{g}(\hat{z}) \geq -x(z - \hat{z}).$$

Let $D_1, D_2 \subset Z$ be bounded domains such that $\sigma \subset D_1 \subset D_2 \subset\subset \gamma$. Next, let $l(z) \in C^\infty(\mathbf{R}^k)$ be supported on the unit ball $B_1 \subset \mathbf{R}^k$ and $\int_{\mathbf{R}^k} l(z)\,dz = 1$ and set

$$l_\delta(z) = \frac{1}{\delta^k} l\left(\frac{z}{\delta}\right).$$

Let $h \in C^\infty(Z), h = 1$ on D_1, $h = 0$ on $Z \setminus D_2, 0 \leq h \leq 1$ on Z. Set

$$g_\epsilon = \widetilde{g} * l_\epsilon,$$

$$G_\epsilon = hg + (1 - h)g_\epsilon.$$

Since the graph of the function g satisfies the λ-cone condition it follows that the upper normals to the graph is in the cone K. Since K^* is a convex cone the upper normals to the graphs of the functions g_ϵ satisfies the 2λ-cone condition for all small $\epsilon > 0$, and hence the graphs of linear function $d_z g_\epsilon$ is in $L_{2\lambda}$ for all z where d_z is the differential at z. Since the functions g_ϵ are defined on the whole space $\mathbf{R}^k$ it follows that the graphs of the functions g_ϵ satisfies the λ-cone condition. Really, let a, b, $a \neq b$ be on the graph of g_ϵ. If $a - b \notin L_{2\lambda}$ then there is a point $\alpha \in [\pi(a), \pi(b)]$ such that $d_\alpha g_\epsilon \notin L_{2\lambda}$. For any $k > 0$ the function $g_\epsilon \to g$ in $C^k(D_2)$ as $\epsilon \to 0$. Hence for a sufficiently small $\epsilon_o > 0$ the graph of the function $G_{\epsilon_0} := G$ will satisfy the 2λ-cone condition. Moreover G will be a smooth function on Z, $G = g$ on D_1 and $|\nabla G| < \sqrt{n}$ on Z.

Let us set

$$F := s - G(z).$$

Denote

$$b := \nabla F = (-\nabla G, 1),$$

$$a := (\nabla G/|\nabla G|, |\nabla G|).$$

The vector a is tangent to the level surface of the function F, and $tr(b) = \sqrt{n}$. Since level surfaces of the function F satisfies the 2λ-cone condition and $a \cdot b = 0$, it follows that $a \in L_{2\lambda}$ and hence $b \in K_{2\lambda}$. Therefore the function F satisfies the ellipticity conditions with the ellipticity constant $\Lambda < 4\lambda^2 \sqrt{n}$.

Remark 1. For a real-analytic manifold Σ one can obtain the existence of a real-analytic function F after insignificant changes in the construction.

Remark 2. The proof of the lemma holds if instead of compactness of Σ we assume that Σ is a smooth closed manifold with a boundary.

Proof of Main Lemma. Set

$$\lambda = (n-1)M$$

Let ξ and η be correspondingly negative and nonnegative subspace of the quadratic form $a - b$ in $\mathbf{R}^n$. Denote by $c \in Q$ the quadratic form $l|\xi|^2 + m|\eta|^2$, $l, m > 0$ such that $(a - b) \cdot c = 0$. Then $1/(n-1)M < l/m < (n-1)M$ by (H) 2) and hence the set $H(S^{n-1})$ satisfies the λ-cone condition.

For $\Sigma = H(S^{n-1})$ we define function F by Lemma. Then the function w satisfies the equation

$$F(D^2 w) = 0.$$

on $\mathbf{R}^n \setminus \{0\}$. We show now that w is a viscosity solution of (1) on the whole space $\mathbf{R}^n$.

Let $p(x)$, $x \in \mathbf{R}^n$ be a quadratic form such that $p \leq w$ on $\mathbf{R}^n$. We choose any quadratic form $p'(x)$ such that $p \leq p' \leq w$ and there is a point $x' \neq 0$ at which $p'(x') = w(x')$. Then it follows that $F(p) \leq F(p') \leq 0$. Consequently for any quadratic form $p(x)$ from the inequality $p \leq w$ ($p \geq w$) it follows that $F(p) \leq 0$ ($F(p) \geq 0$). This implies that w is a viscosity solution of (1) in $\mathbf{R}^n$ (see Proposition 2.4 in [CC]).

3 Cubic form P

In this section we introduce and investigate the cubic form which will be used to construct our non-classical solutions. Let $V = (X, Y, Z) \in \mathbf{R}^{12}$ be a variable vector with X, Y, and $Z \in \mathbf{R}^4$. For any $t = (t_0, t_1, t_2, t_3) \in \mathbf{R}^4$ we denote by $qt = t_0 + t_1 \cdot i + t_2 \cdot j + t_3 \cdot k \in \mathbf{H}$ (Hamilton quaternions). For any $q = q_0 + q_1 \cdot i + q_2 \cdot j + q_3 \cdot k \in \mathbf{H}$ its conjugate will be denoted $q^* = q_0 - q_1 \cdot i - q_2 \cdot j - q_3 \cdot k$; thus, $q^* q = q q^* = |q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$.

Define the cubic form $P = P(V) = P(X, Y, Z)$ as follows

$$P(X, Y, Z) = Re(qX \cdot qY \cdot qZ) = X_0 Y_0 Z_0 - X_0 Y_1 Z_1 - X_0 Y_2 Z_2 - X_0 Y_3 Z_3$$

$$-X_1 Y_0 Z_1 - X_1 Y_1 Z_0 - X_1 Y_2 Z_3 + X_1 Y_3 Z_2 - X_2 Y_0 Z_2 + X_2 Y_1 Z_3 - X_2 Y_2 Z_0 - X_2 Y_3 Z_1$$

$$-X_3Y_0Z_3 - X_3Y_1Z_2 + X_3Y_2Z_1 - X_3Y_3Z_0.$$

Let $d = (a, b, c) \in \mathbf{R}^{12}$ be a vector with the norm $\sqrt{3}$, $|| a ||^2 + || b ||^2 + || c ||^2 = 3$. Define the quadratic form

$$Q_d = Q_{a,b,c} = Q_{a,b,c}(X, Y, Z)$$

by differentiating P in the direction d:

$$Q_{a,b,c}(X, Y, Z) = \sum_{i=0}^{4} a_i \partial P/\partial X_i + \sum_{i=0}^{4} b_i \partial P/\partial Y_i + \sum_{i=0}^{4} c_i \partial P/\partial Z_i \ .$$

A direct calculation shows that

$$Q_d(X, Y, Z) = X^t M_c Y + X^t M_b^t Z + Y^t M_a Z$$

where, in general, we define the matrix M_s for an arbitrary $s \in \mathbf{R}^4$ by

$$M_s = \begin{pmatrix} s_0 & -s_1 & -s_2 & -s_3 \\ -s_1 & -s_0 & -s_3 & s_2 \\ -s_2 & s_3 & -s_0 & -s_1 \\ -s_3 & -s_2 & s_1 & -s_0 \end{pmatrix}$$

Direct (and easy) calculations show that M_s has the following properties:

1). $\qquad M_s \cdot M_s^t = M_s^t \cdot M_s = || s ||^2 I_4$;

thus, M_s is proportional to an orthogonal matrix. In particular, if $|| s || = 1$ then M_s is orthogonal itself. In general, we write $M_s = || s || O_s$ with $O_s \in O(4)$.

2). $\det(M_s) = - || s ||^4, \qquad \det(O_s) = -1$;

3). the characteristic polynomial $PM_s(x)$ of M_s factors as

$$PM_s(x) = (x^2 - || s ||^2)(x^2 + 2s_0 x + || s ||^2)$$

with $s_0 = Re(qs)$; and that of O_s as

$$PO_s(x) = (x^2 - 1)(x^2 + 2s_0^* x + 1)$$

with $s_0^* = s_0/ | qs | = Re(qs/ | qs |)$;

4). define the symmetric matrix $N_s = (O_s + O_s^t)$; then its characteristic polynomial $PN_s(x) = (x^2 - 4)(x + 2s_0^*)^2$, its spectrum being $Sp(N_s) = \{2, -2, -2s_0^*, -2s_0^*\}$;

5). M_s is the matrix (with respect to the standard basis) of the endomorphism $\mathbf{H} \to \mathbf{H}$, $q \mapsto \bar{q} \cdot \bar{q} s$.

The points 3 and 5 applied to the product matrix $M_{rst} = M_r \cdot M_s \cdot M_t$, r, s, t being arbitrary vectors in $\mathbf{R}^4$ give the following formula for the characteristic polynomial PM_{rst} of M_{rst}:

$$PM_{rst}(x) = (x^2 - \parallel r \parallel^2 \parallel s \parallel^2 \parallel t \parallel^2)(x^2 + 2P(r,s,t)x + \parallel r \parallel^2 \parallel s \parallel^2 \parallel t \parallel^2)$$

with $P(r,s,t) = Re(qr \cdot qs \cdot qt)$ as above. Indeed, M_{rst} is conjugate to the matrix of the endomorphism $q \mapsto \bar{q} \cdot \bar{q} r \cdot \bar{q} s \cdot \bar{q} t$.

For the corresponding orthogonal matrix O_{rst} we get the polynomial

$$PO_{rst}(x) = (x^2 - 1)(x^2 + 2\bar{P}(r,s,t)x + 1)$$

where $\bar{P}(r,s,t) = P(r,s,t)/(\parallel r \parallel \cdot \parallel s \parallel \cdot \parallel t \parallel)$ and for the corresponding symmetric matrix $N_{rst} = O_{rst} + O_{rst}^t$ the spectrum is

$$(*) \quad Sp(N_{rst}) = \{2, -2, \ -2\bar{P}(r,s,t), -2\bar{P}(r,s,t)\}.$$

Warning: in the case of the product of two matrices M_r, M_s the characteristic polynomial is completely different; namely, if $M_{rs} = M_r \cdot M_s$ then

$$PM_{rs}(x) = (x^2 - 2(r,s)x + \parallel r \parallel^2 \parallel s \parallel^2)^2$$

with the usual scalar product (r,s).

Define now two quantities $m = m(d) = m(a,b,c) = \parallel a \parallel \cdot \parallel b \parallel \cdot \parallel c \parallel$, $n = n(d) = n(a,b,c) = P(a,b,c)$. Clearly, $\mid n(d) \mid \leq m(d) \leq 1$ by the inequality between the geometric and quadratic means, since $\parallel a \parallel^2 + \parallel b \parallel^2 + \parallel c \parallel^2 = 3$.

Proposition 1. *The characteristic polynomial $CH_d(x)$ of the quadratic form $2Q_d$ equals*

$$CH_d(x) = (x^3 - 3x + 2m)(x^3 - 3x - 2m)(x^3 - 3x + 2n)^2$$

Proof. We have

$$Q_d(X,Y,Z) = X^t M_c Y + X^t M_b^t Z + Y^t M_a Z$$

$$= \parallel c \parallel X^t O_c Y + \parallel b \parallel X^t O_b^t Z + \parallel a \parallel Y^t O_a Z.$$

Let us perform the orthogonal change of variables given by:

$$x = O_c^t X, \quad y = Y, \quad z = O_a Z.$$

Then in these new variables the form Q_d becomes equal to

$$\tilde{Q}_d(x,y,z) = \parallel c \parallel x^t y + \parallel b \parallel x^t O_c^t \cdot O_b^t \cdot O_a^t z + \parallel a \parallel y^t z.$$

Thus, the matrix of the form $2\tilde{Q}_d$ is the following block matrix:

$$\tilde{M}_d = \begin{pmatrix} 0_4 & \| \, c \, \| \, I_4 & \| \, b \, \| \, O_{abc}^t \\ \| \, c \, \| \, I_4 & 0_4 & \| \, a \, \| \, I_4 \\ \| \, b \, \| \, O_{abc} & \| \, a \, \| \, I_4 & 0_4 \end{pmatrix}$$

where 0_4 and I_4 are the zero and the unit 4x4 matrices, respectively, $O_{abc} = O_a \cdot O_b \cdot O_c$ as above.

Let now $\lambda \in Sp(\tilde{M}_d)$, $v_\lambda = (x_\lambda, y_\lambda, z_\lambda)$ being a corresponding eigenvector, normalized by the conditions $\| \, v_\lambda \, \| = \sqrt{3}$, $(v_\lambda, d) \geq 0$.
The condition $\tilde{M}_d \cdot v_\lambda = \lambda v_\lambda$ gives

$$\lambda x_\lambda = \| \, c \, \| \, y_\lambda + \| \, b \, \| \, O_{abc}^t z_\lambda$$

$$\lambda y_\lambda = \| \, c \, \| \, x_\lambda \quad + \quad \| \, a \, \| \, z_\lambda$$

$$\lambda z_\lambda = \| \, b \, \| \, O_{abc} x_\lambda + \| \, a \, \| \, y_\lambda \qquad .$$

Multiplying the second and the third equations by λ and inserting in thus obtained equations the first one one finds

$$(\lambda^2 - \| \, c \, \|^2) x_\lambda = (\| \, a \, \| \cdot \| \, c \, \| + \lambda \| \, b \, \| \, O_{abc}^t) z_\lambda$$

$$(\lambda^2 - \| \, a \, \|^2) z_\lambda = (\| \, a \, \| \cdot \| \, c \, \| + \lambda \| \, b \, \| \, O_{abc}) x_\lambda$$

which implies

$$(\lambda^2 - \| \, a \, \|^2)(\lambda^2 - \| \, c \, \|^2) x_\lambda = (\| \, a \, \| \cdot \| \, c \, \| + \lambda \| \, b \, \| \, O_{abc}^t)(\| \, a \, \| \cdot \| \, c \, \| + \lambda \| \, b \, \| \, O_{abc}) x_\lambda$$

and, after simplifying,

$$\lambda(\lambda^3 I_4 - 3\lambda I_4 - m N_{abc}) x_\lambda = 0,$$

since $\| \, a \, \|^2 + \| \, b \, \|^2 + \| \, c \, \|^2 = 3$, $m = \| \, a \, \| \cdot \| \, b \, \| \cdot \| \, c \, \|$, $O_{abc} O_{abc}^t = I_4$, $N_{abc} = O_{abc} + O_{abc}^t$.
Hence, either $\lambda = 0$ or

$$(\lambda^3 - 3\lambda) \in m \cdot Sp(N_{abc}) = \{-2m, 2m, -2n, -2n\}.$$

This finishes the proof for $\lambda \neq 0$. If $\lambda = 0$ we get the conditions

$$0 = \| \, c \, \| \, y_\lambda + \| \, b \, \| \, O_{abc}^t z_\lambda$$

$$0 = \| \, c \, \| \, x_\lambda \quad + \quad \| \, a \, \| \, z_\lambda$$

$$0 = \| \, b \, \| \, O_{abc} x_\lambda + \| \, a \, \| \, y_\lambda \qquad .$$

immediately implying that $m = 0$ (since else these equations give $x_\lambda = 0$) and the formula holds for this case as well.

Corollary 1. *Define the angles $\alpha, \beta \in [0, \pi]$ by $m = \cos\alpha, n = \cos\beta$. Then*

$$Sp(\tilde{M}_d) = \{2\cos(\alpha/3 + \pi k/3), 2\cos(\beta/3 + \pi(2l+1)/3), 2\cos(\beta/3 + \pi(2l+1)/3)\},$$

$$k = 0, 1, \ldots 5, \quad l = 0, 1, 2.$$

Proof. Indeed, if we put $\lambda = 2\cos\gamma$, the equations $\lambda^3 - 3\lambda + 2m = 0$, $\lambda^3 - 3\lambda - 2m = 0$ and $\lambda^3 - 3\lambda + 2n = 0$ become respectively, $\cos(3\gamma) = -\cos\alpha$, $\cos(3\gamma) = \cos\alpha$ and $\cos(3\gamma) = -\cos\beta$ which implies the result.

Let us now order the eigenvalues in the decreasing order:

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5 \geq \lambda_6 \geq \lambda_7 \geq \lambda_8 \geq \lambda_9 \geq \lambda_{10} \geq \lambda_{11} \geq \lambda_{12}.$$

Corollary 2.
1). $2 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq 1$;
2). $-1 \geq \lambda_9 \geq \lambda_{10} \geq \lambda_{11} \geq \lambda_{12} \geq -2$;
3). $\lambda_1 \geq \sqrt{3}$; $\lambda_{12} \leq -\sqrt{3}$;
4). *If $\lambda_1/\lambda_3 = 2$ (resp. $\lambda_{12}/\lambda_{10} = 2$) then the polynomial $CH_d(x) = (x+2)^3(x-2)(x+1)^2(x-1)^6$ (resp. $CH_d(x) = (x-2)^3(x+2)(x-1)^2(x+1)^6$), and $d = v_1$ (resp. $d = v_{12}$) where v_i is the normalized eigenvector corresponding to λ_i.*

Proof. All these conclusions, except that concerning v_1 (resp. v_{12}) follow from Corollary 1 along with the following elementary lemma:

Lemma. *Let $F_m(x) = (x^3 - 3x - 2m)$ with $\mid m \mid \leq 1$, and let $x_1 \geq x_2 \geq x_3$ be its roots.*
1). *If $0 \leq m \leq 1$ then $2 \geq x_1 \geq \sqrt{3}$, $-1 \geq x_3$, and each of the conditions $2 = x_1$, $-1 = x_3$ implies $m = 1$.*
2). *If $0 \geq m \geq -1$ then $-2 \leq x_3 \leq -\sqrt{3}$, $1 \leq x_1$, and each of the conditions $1 = x_1$, $-2 = x_3$ implies $m = -1$.*
3). *If $\mid m \mid \leq 0.75$ then $\mid x_1 \mid > 1.38$, $\mid x_3 \mid > 1.38$.*
This lemma follows from the monotonicity of $\cos(x)$ on $[0, \pi]$ along with the inequalities $2\cos(\frac{\arccos(.75)+2\pi}{3}) < -1.38$, $2\cos(\frac{\arccos(.75)}{3}) > 1.94$.
To prove that $d = v_1$ one notes that $\lambda_1/\lambda_3 = 2$ implies $m(d) = n(d) = 1$ which means the function P has an absolute maximum at d, its derivative in the direction d equals 1 which means that $2Q_d(d) = 2$, i.e. $d = v_1$. The case of v_{12} is completely similar.

Corollary 3. *Define*

$$\delta = \sup_{d, ||d|| = \sqrt{3}} \{\max\{\frac{\lambda_+^\perp(d)}{\lambda_3(d)}, \quad \frac{\lambda_-^\perp(d)}{\lambda_{10}(d)}\}\}$$

where

$$\lambda_+^\perp(d) = 2 \sup_{v \perp d,\, ||v||=\sqrt3} Q_d(v), \quad \lambda_-^\perp(d) = 2 \inf_{v \perp d,\, ||v||=\sqrt3} Q_d(v).$$

Then $\delta < 3/2$.

Proof. By Lemma, part 3 it is true for $\mid n \mid\,\le 0.75$ since $2/1.38 < 3/2$. Let now $n \ge 0.75$ (the case $n \le -0.75$ being symmetric). Suppose that $\frac{\lambda_+^\perp(d)}{\lambda_3(d)} \ge 3/2$, and hence $\lambda_+^\perp(d) \ge 1.5$ (since $\lambda_3(d) \ge 1$). We will show that the conditions $\lambda_+^\perp(d) \ge 1.5$ and $n \ge 0.75$ are incompatible. Indeed, define

$$T(x,y) = P(xd + y\sqrt3 v_+^\perp(d)) = t_3 x^3 + t_2 x^2 y + t_1 xy^2 + t_0 y^3$$

where $v_+^\perp(d)$ is a norm $\sqrt3$ vector on which $Q_d(v)$ achieves the maximum. We get that $t_3 = T(1,0) = P(d) = n \ge 0.75$, $t_1 = T_x(0,1) = 3\lambda_+^\perp(d)/2 \ge 9/4$. For any (x,y) with $x^2 + y^2 = 1$, $\mid T(x,y) \mid = \mid P(xd + y\sqrt3 v_+^\perp(d)) \mid\le 1$. Let now $x_0 = \frac{1}{\sqrt2}$, $y_0 = \pm\frac{1}{\sqrt2}$ where the sign of y_0 is chosen from the condition that $y_0 \frac{t_0+t_2}{\sqrt2} \ge 0$. Then $\mid T(x_0, y_0) \mid\ge \frac{t_3+t_1}{2\sqrt2} \ge \frac{3}{2\sqrt2} > 1$ which is a contradiction.

The following result will be used in Section 4 to deduce our main result.

Corollary 4. *Let $u \ne v \in S^{11}_{\sqrt3}$ be two vectors of norm $\sqrt3$. Then*

$$P(u) - P(v) \le 3\sqrt3 \lambda_3(d) \mid u - v \mid /4.$$

$$P(u) - P(v) \ge 3\sqrt3 \lambda_{10}(d) \mid u - v \mid /4.$$

where $d = \sqrt3 \frac{u-v}{|u-v|}$.

Proof. Denote $s = \mid u-v \mid /2$, $z = d^\perp \bigcap [u, v]$. Writing the Taylor development for the (cubic) function P, we get

$$P(u) - P(v) = 2s(Q_d(z)/\sqrt3 + s^2 P_{ddd}/18\sqrt3).$$

Since

$$Q_d(z) \le (1-s^2/3) \sup_{x \perp d,\, ||x||=\sqrt3} Q_d(x) \le 3(1-s^2/3)\lambda_+^\perp(d)/2 \le 9(1-s^2/3)\lambda_3(d)/4,$$

$P_{ddd} \le 2/\sqrt3 \le 2\lambda_3(d)/\sqrt3$ we get

$$P(u) - P(v) \le 2s\lambda_3(d)(9(1-s^2/3)/4\sqrt3 + s^2/27) \le 3\sqrt3\lambda_3(d) \mid u - v \mid /4.$$

The proof of the second inequality is completely similar.

Remark.   Let us resume the spectral properties of $2Q_d$ when d varies over $S = S^{11}_{\sqrt3}$. We have a stratification $V_0 \subset S \supset T \supset V = V_+ \bigcup V_-$ where $T =$

$S_1^3 \times S_1^3 \times S_1^3$ is defined by the condition $m(d) = 1$, V_+ (resp. V_-) is defined by $P(d) = 1$ (resp. $P(d) = -1$), $V_0 = \{d : m(d) = 0\}$; each of V_+ and V_- is diffeomorphic to $S_1^3 \times S_1^3$. On V_+ (resp. V_-) we have the characteristic polynomial $(x+2)^3(x-2)(x+1)^2(x-1)^6$ (resp. $(x-2)^3(x+2)(x-1)^2(x+1)^6$); on $S \setminus (T \bigcup V_0)$ we have $\sqrt{3} < \lambda_1(d) < 2$, $1 < \lambda_4(d)$, $-1 > \lambda_9(d)$, $-\sqrt{3} > \lambda_{12}(d) > -2$. Finally, on V_0 the polynomial equals $x^4(x^2 - 3)^4$.

4 Function w and map H

In this section we show that the function

$$w(x) = P(x)/ \mid x \mid$$

is what we want, i. e. the map

$$H : S_1^{11} \longrightarrow Q, \quad H(a) = Hess(w(a))$$

verifies the condition (H) of Section 2.

Proposition 2. Let $a \neq b \in S_1^{11}$. Then there exist two vectors $e, f \in S_1^{11}$ such that

$$w_{ee}(a) - w_{ee}(b) \geq \quad \mid a - b \mid /4\sqrt{3},$$
$$w_{ff}(a) - w_{ff}(b) \leq - \mid a - b \mid /4\sqrt{3}.$$

Proof. Let $d = \sqrt{3}\frac{a-b}{|a-b|}$. Recall that we denote by v_i, $i = 1, \ldots, 12$ the normalized eigenvectors of the form $2Q_d$ from Section 3 (the eigenvalues $\lambda_i = \lambda_i(d)$ being ordered in the decreasing order). Let V^+ be the 3-dimensional space generated by v_1, v_2, v_3 and let $e \in S_1^{11} \bigcap V^+ \bigcap a^\perp \bigcap b^\perp$. It means in particular that $2Q_d(e) \geq \lambda_3(d)$. The conditions $b \perp e$, $a \perp e$ imply

$$w_{ee}(a) = P_{ee}(a) - P(a), \; w_{ee}(b) = P_{ee}(b) - P(b),$$

hence

$$w_{ee}(a) - w_{ee}(b) = P_{ee}(a) - P_{ee}(b) - (P(a) - P(b)).$$

Since $P_{ee}(x)$ is a linear function we get

$$P_{ee}(a) - P_{ee}(b) = \mid a - b \mid P_{eed}/\sqrt{3} = 2 \mid a - b \mid Q_d(e)/\sqrt{3} \geq \lambda_3(d) \mid a - b \mid /\sqrt{3}.$$

Now,

$$P(a) - P(b) = \frac{(P(a\sqrt{3}) - P(b\sqrt{3}))}{3\sqrt{3}} \leq \frac{3\sqrt{3}\lambda_3(d)\sqrt{3} \mid a - b \mid}{12\sqrt{3}} = \frac{\sqrt{3}\lambda_3(d) \mid a - b \mid}{4}$$

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by Corollary 4, and we get

$$w_{ee}(a) - w_{ee}(b) \geq \lambda_3(d) \mid a - b \mid (\frac{1}{\sqrt{3}} - \frac{\sqrt{3}}{4}) \geq \frac{\mid a - b \mid}{4\sqrt{3}} .$$

The second inequality is proven replacing e by $f \in S^{11} \bigcap V^- \bigcap a^\perp \bigcap b^\perp$ where V^- is generated by v_{10}, v_{11}, v_{12}.

Corollary 5.

1). *The map* $H : S_1^{11} \longrightarrow Q, \quad H(a) = Hess(w(a))$ *is a smooth embedding.*

2). *Let for* $a \neq b \in S_1^{11}$

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{11} \geq \mu_{12}$$

be the eigenvalues of $Hess(w(a)) - Hess(w(b))$. *Then*

$$M^{-1} = \frac{1}{1536\sqrt{3}} \leq -\frac{\mu_1}{\mu_{12}} \leq 1536\sqrt{3} = M.$$

Proof.

1). This follows immediately from Proposition 2.

2). An easy calculation shows that $\mid w_{efg}(x) \mid \leq 32$ for any $e, f, g, x \in S_1^{11}$. Hence

$$\mid w_{ef}(a) - w_{ef}(b) \mid \leq \mid w_{efd'}(d') \mid \cdot \mid a - b \mid \leq 32 \mid a - b \mid$$

for $d' = d/\sqrt{3}$. Since all elements of the matrix $Hess(w(a)) - Hess(w(b))$ are of absolute value $\leq 32 \mid a - b \mid$, all its eigenvalues are of absolute value $\leq 12 \cdot 32 \mid a - b \mid$. Using the inequalities of Proposition 2 we get the conclusion.

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